Discrete Mathematics Quiz 3 solutions

Question 1: How many ways are there to write 1000000 as the product of 3 factors if order of factors matter?

Solution: Suppose $1000000 = p \times q \times r$ generically, then we have to count the number of possible combinations of (p, q, r). We have $p = 2^a 5^x, q = 2^b 5^y, r = 2^c 5^z$ such that $a + b + c = x + y + z = 6$. Each combination of nonnegative solutions (a, b, c) and (x, y, z) to $a + b + c = 6$ and $x + y + z = 6$ gives rise to a unique (p, q, r)(and vice versa, which means there is a bijection) such that product is 1000000. The number of such combinations is therefore $\binom{8}{2}^2 = 784$. ■

Question 2: How many ways are there to place 2 rooks in the chessboard such that they don't attack each other?

Solution: The first rook can be placed in any of the 64 squares. No matter where it is placed, it covers 15 other squares including itself. That is, the second rook can be placed anywhere on the board except the 15 squares in the same row or column as the first rook. So, it has $64 - 15 = 49$ possible positions. So, the total number of ways to place two rooks is $64 \times 49 = 3136$. In case the rooks are identical, we just have to divide this number by 2 because each combination will be counted twice in this way but since there was no such option, we can safely assume that the rooks were not identical. ■

Question 3: How many 9-digit numbers are there such that there are 5 distinct digits and one digit occurs 5 times?

Solution: This problem requires case by case analysis. Consider the first case when the most significant digit of the number does not occur 5 times, i.e. it occurs exactly once. Now, from among the 8 remaining positions, we choose the 5 positions where the repeated digit occurs in $\binom{8}{5} = 56$ ways. After choosing the positions, the first digit(most significant digit) can be chosen in 9 ways(since it cannot be 0) and the second distinct digit can be chosen in 9 ways since it cannot be equal to the first digit. The third distinct digit can be chosen in 8 ways since it cannot be equal to the first two chosen digits and so on. Therefore, the number of ways to choose the digits is $9 \times 9 \times 8 \times 7 \times 6 = 27216$. This gives rise to a total of $27216 \times 56 = 1524096$ such numbers. Consider the other case when the most significant digit occurs 5 times. Then, the remaining 4 positions can be chosen in $\binom{8}{4} = 70$ ways. After choosing the positions, we should choose the digits. The most significant digit has 9 possibilities as it cannot be 0. This fixes the digits in four other positions. Now, the other distinct digits can be chosen in $9 \times 8 \times 7 \times 6$ ways similarly. So, total number of ways of choosing the digits is $9 \times 9 \times 8 \times 7 \times 6 = 27216$ like the previous case. The total number of numbers that this gives rise to is, therefore, $27216 \times 70 = 1905120$. Finally, we get that there are $1524096 + 1905120 = 3429216$ such 9-digit numbers. ■

Question 4: How many ways can a person choose from 5 distinct pencils, 6 distinct pens and 8 distinct erasers if he has to buy at most one pencil, at most one pen and at most 1 eraser?

Solution: From among pencils, he has got 6 possibilities. He can either buy no pencil or buy exactly one pencil from among the 5 distinct pencils. Similarly, from among pens, he can either buy no pen or buy any of the 6 distinct pens thereby giving him 7 possibilities. Similarly, he has 9 possibilities to buy erasers. Since they are mutually independent, the total number of ways to buy the items is $5 \times 7 \times 9 = 378$. ■

Question 5: How many 8-digit numbers have all digits odd or all digits even?

Solution: When all digits are odd, each digit can be one of $1, 3, 5, 7, 9$ and hence, has 5 possibilities. So, there are 5^8 such numbers. When all digits are even, each digit can be $0, 2, 4, 6, 8$ and hence has 5 possibilities but the first digit cannot be 0 and so, it has only 4 possibilities. There are 4×5^7 such numbers. The total number of such number is therefore, $5^8 + 4 \times 5^7 = 703125$. ■

Question 6: How many integer solutions are there to the equation $a_1 + a_2 + \ldots + a_{10} = 75$ such that $a_i \geq i$ for all i?

Solution: Let $b_i = a_i - i$. Then, each solution $(a_1, \ldots, a_{10})$ of the above system corresponds uniquely to a solution $(b_1, \ldots, b_{10})$ of the system $b_1 + \ldots + b_{10} = 75 - 55 = 20$ such that $b_i \geq 0$(because $a_i \geq i \implies b_i = a_i - i \geq 0$) for all i. And each solution $(b_1, \ldots, b_{10})$ of the second system corresponds to a unique solution of the first system because $b_i \geq 0 \implies a_i = b_i + i \geq i$. So, there is a bijection and the number of solutions to the given system is equal to the number of solutions to the second system which is just $\binom{29}{9}$. ■

Question 7: How many ways are there to choose 4 cards of different suits and different values from a standard deck of 52 cards?

Solution: Since the 4 cards contain different suits, they contain all the suits. Suppose the cards are a_1 of clubs, a_2 of diamonds, a_3 of hearts and a_4 of spades. Then, we have $1 \leq a_i \leq 13$(where we denote Ace, King, Queen, Jack by $1, 11, 12, 13$ respectively) for each i and since no two cards have the same value, we have that a_1, a_2, a_3, a_4 are pairwise distinct. Then, there is a bijection between all the required choices of the 4 cards and (a_1, a_2, a_3, a_4) such that $1 \leq a_i \leq 13$ and a_1, a_2, a_3, a_4 are pairwise distinct. The number of such choices for (a_1, a_2, a_3, a_4) is clearly $13 \times 12 \times 11 \times 10 = 17160$. ■

Question 8: How many ways are there to choose 6 cards from a standard deck of 52 cards such that all 4 suits are present?

Solution: Assume for the sake of brevity that the suits are A, B, C, D. Let x, y, z, w be the number of cards chosen in the suits A, B, C, D respectively. Then, we would like to have $x, y, z, w \geq 1$ and $x+y+z+w = 6$. All possible cases for (x, y, z, w) are $(3, 1, 1, 1), (2, 2, 1, 1)$ and their permutations. Number of ways to choose 6 cards so as to get $(x, y, z, w) = (3, 1, 1, 1)$ is $\binom{13}{3} \times \binom{13}{1} \times \binom{13}{1} \times \binom{13}{1} = 628342$. Since there are 4 permutations of $(3, 1, 1, 1)$, we get a total of $628342 \times 4 = 2513368$ choices. We can choose 6 cards so as to get $(x, y, z, w) = (2, 2, 1, 1)$ in $\binom{13}{2} \times \binom{13}{2} \times \binom{13}{1} \times \binom{13}{1} = 1028196$ ways. Since there are 6 permutations of $(2, 2, 1, 1)$, we get a total of $1028196 \times 6 = 6169176$ choices. So, the total number of ways to choose 6 cards is $2513368 + 6169176 = 8682544$. ■

Question 9: How many ways can n boys and n girls be arranged in a line such that no two boys are consecutive and no two girls are consecutive?

Solution: Since no two boys are consecutive and no two girls are consecutive, the arrangement has to be either $BGBG \dots BG$ or $GBGB \dots GB$ where B denotes boy and G denotes girl. In either of these arrangements, the number of ways to arrange the boys and girls in this order is $(n!)^2$ because among the n positions allotted for boys, they can permute among themselves in $n!$ ways and for all these permutations, among the positions allotted for girls, they can permute among themselves in $n!$ ways giving a total of $(n!)^2$ ways. Since this holds for either of these arrangements, the total number of arrangements is therefore $2(n!)^2$. ■

Question 10: If 13 books are arranged in a row, how many ways are there to choose 5 books such that no two adjacent books are chosen?

Solution: Let x_0 be the number of books before the first chosen book. Let x_1 be the number of books between the first chosen book and the second chosen book. Define x_2, x_3, x_4, x_5 analogously. So, x_5 will be the number of books after the last chosen book. Then, we have $x_0 + x_1 + \dots + x_5 = 8$. Since no two adjacent books are chosen, we have $x_1, x_2, x_3, x_4 \geq 1$ and $x_0, x_5 \geq 0$. There is a bijection between choosing books and solutions to the above equation. If we define $y_0 = x_0, y_1 = x_1 - 1, y_2 = x_2 - 1, y_3 = x_3 - 1, y_4 = x_4 - 1, y_5 = x_5$, then we are looking for nonnegative integer solutions to $y_0 + \dots + y_5 = 4$. There are $\binom{9}{5} = 126$ solutions and this is the answer to the question. ■

Question 11: How many 10-digit numbers are there such that the sum of the digits is at most 88?

Solution: The total number of 10-digit numbers is 9000000000. The sum of 10 digits in a 10-digit number is at most $9 \times 10 = 90$. Now, the number of 10-digit numbers with sum of digits equal to 90 is exactly 1, namely from the number 9999999999. We count the number of 10-digit numbers with sum of digits equal to 89. This can be achieved only if the digits contain exactly one 8 and nine 9s. The number of such numbers is then 10 because the 8 can occur in any of the 10 digits. So, the number of required 10-digit numbers is $9000000000 - 1 - 10 = 8999999989$. ■

Question 12: How many 5-digit numbers with middle digit 7 are there such that it is divisible by 3?

Solution: If we denote the number generically as $\overline{ab7cd}$, then since it is divisible by 3, we have that $a + b + 7 + c + d$ is divisible by 3 and so, $a + b + c + d$ leaves remainder 2 modulo 3 and so, the 4-digit number $\overline{abcd}$ leaves remainder 2 modulo 3. It is easy to see that there is a bijection between all such required 5-digit numbers and 4-digit numbers which leave remainder 2 modulo 3. The latter is just counting $1001, 1004, \ldots, 9998$ giving a total of 3000 such numbers. ■

Question 13: How many non-injective functions are there from the set $\{1, 2, \ldots, n\}$ to itself?

Solution: The total number of functions is n^n. The total number of injective functions is $n!$. So, the total number of non-injective functions is $n^n - n!$. ■

Question 14: How many n-digit numbers are there such that no two adjacent digits are equal?

Solution: The first digit can be chosen in 9 ways because it can be any of $1, 2, \ldots, 9$. It cannot be 0 because then, the number will not be a n-digit number. Suppose we fix the first digit. Now, the second digit can be any of $0, 1, \ldots, 9$ except for the first digit. So, it has 9 possibilities. Once we fix the first two digits, the third digit can be any digit except the second digit because adjacent digits should not be equal. So, the third digit has 9 possibilities. This continues until the last digit. So, the total number of such n-digit numbers that we are looking for is 9^n. ■

Question 15: How many ways can we place 8 identical rooks on the chessboard such that no two attack each other?

Solution: If two rooks are located in the same row, then they will attack each other. So, no two rooks are located in the same row. Since there are 8 rooks and 8 rows, each row contains exactly one rook. Suppose the rooks are located in the cells $(1, i_1), (2, i_2), \ldots, (8, i_8)$. Then, since no two rooks attack each other, we have that the columns $i_1, \ldots, i_8$ are all distinct. Since there are only 8 columns, $i_1, \ldots, i_8$ is a permutation of $1, 2, \ldots, 8$. Note finally that each permutation $i_1, \ldots, i_8$ of $1, 2, \ldots, 8$ gives rise to a unique arrangement of 8 rooks on the chessboard such that no two attack each other, namely the arrangement $(1, i_1), (2, i_2), \ldots, (8, i_8)$. So, because of this bijection, the number of ways to place 8 rooks is just the number of permutations of $1, 2, \ldots, 8$ which is $8! = 40320$. ■

Question 16: How many permutations of the letters of the word "raffled" are there such that vowels are NOT together?

Solution: The total number of permutations of the word "raffled" is $\frac{7!}{2!}$ because there are two $'f'$s. Now, the number of permutations such that the vowels are together is $2 \times \frac{6!}{2!}$ because if the vowels are together, they can be either "ae" or "ea" and then, we consider them as one block. So, there are then 6 letters(one of them being the vowel block) and two $'f'$s which can be permuted in $\frac{6!}{2!}$ ways. Since this happens for "ae" as well as "ea", the total number of permutations where the vowels are together is $2 \times \frac{6!}{2!}$. The answer is $\frac{7!}{2!} - 2 \times \frac{6!}{2!} = 1800$. ■

Question 17: For positive integers m, n, how many ways are there to go from the point $(0, 0)$ to the point (m, n) if in each step, we can either go one unit right or one unit up?

Solution: For each path, write down a sequence of letters R and U such that, once it starts from $(0, 0)$, everytime the path goes 1 unit right, write down a R and if it goes 1 unit up, write down a U. So, the resulting string that we write has m occurences of R and n occurences of U in some order. Note that for every string that we can write using m, n occurences of R, U respectively, we can get a corresponding path simply by tracing out the path following the letters. For example $RR \dots RUU \dots U$ is the path which goes right m units and then goes up n units. So, there is a bijection between the paths and strings of length $m + n$ containing exactly m occurences of R and exactly n occurences of U. There are $\binom{m+n}{m}$ such strings and this is the answer to the question. ■

Question 18: A company has 15 males and 12 females. How many ways are there to form a team of 5 people such that it contains at least one male and at least one female?

Solution: The total number of people is 27. So, total number of teams of 5 people possible is $\binom{27}{5}$. Among this, the number of teams of 5 people consisting of all males is $\binom{15}{5}$ because there are 15 males and we want to choose a team of 5 males. Similarly, the number of teams consisting of all females is $\binom{13}{5}$. So, total number of teams such that there is at least one male and at least one female is the total number of teams minus the total number of teams with all males minus the total number of teams with all females which is $\binom{27}{5} - \binom{15}{5} - \binom{13}{5}$. ■

Question 19: How many ways can a person visit 3 cities, each of them 4 times such that he starts and ends at different cities?

Solution: If we denote the 3 cities as A, B, C, then we wish to count the number of permutations of $AAAABBBBCCCC$ such that the starting and ending letters are distinct. The number of permutations starting at A and ending at B is $\frac{10!}{3!3!4!} = 4200$. We get the same number for starting and ending letters being $(B, A), (A, C), (C, A), (B, C), (C, B)$ and this counts all the required permutations. So, the answer is $4200 \times 6 = 25200$. ■

Question 20: How many ways are there to choose a football team of 11 students and a volleyball team of 6 students from among 20 students if no student can be in both teams?

Solution: We first choose the football team which can be done in $\binom{20}{11}$ ways. Now, remove the football team and from among the remaining $20 - 11 = 9$ people, we choose the volleyball team of 6 people which can be done in $\binom{9}{6}$ ways. This ensures that no student will be in both teams because we used only the remaining people. This gives a total of $\binom{20}{11}\binom{9}{6}$ ways to pick the teams. ■